Multi-Leader-Follower games without Nash equilibrium interactions

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- Professor in Applied Mathematics at Univ. of Perpignan

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Some personal data

- Professor in Applied Mathematics at Univ. of Perpignan
- **Research topics:**
    - Bilevel programming, Nash games and in particular Multi-leader-follower games
    - Energy management:
        - Electricity markets
        - Industrial Eco-Parks (IEP)
        - Demand-side management
        - and others....(management of renewable energy plants)
    - Variational and quasi-variational inequalities
    - Quasiconvex optimization
- Research lab.: PROMES (CNRS)

Outline of the presentation

- *Multi-Leader-Follower games: a bit of vocabulary*
- *Are they really well-posed?*
- *When does a solution exists? And in which sense?*
- *Single-Leader-Multi-Follower games*
  - *Classical setting (existence, computation)*
  - *SLMF games with uncertainties*
  - *SLMF games without partial coalitions*
  - *SLSF with nonself constraint maps*

Starting point...

Generalized Nash game (GNEP):

$$\boxed{\boxed{\begin{array}{ll}\min\limits_{x_1} & \theta_1(x_1,x_{-1})\\ \text{s.t.} & \left\{\ x_1\in K_1(x_{-1})\right.\end{array}} \quad \cdots \quad \boxed{\begin{array}{ll}\min\limits_{x_n} & \theta_n(x_n,x_{-n})\\ \text{s.t.} & \left\{\ x_n\in K_n(x_{-n})\right.\end{array}}}$$

Starting point...

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So we have n players and they are interacting in a **non cooperative way** (join venture is forbidden or impossible...)

$\bar{x}$ is a Generalized Nash Equilibrium
if and only if
in case a player i would decide to unilaterally deviate from $\bar{x}_i$
(say to $\tilde{x}_i$) then

"he will loose" := "$\theta_i(\tilde{x}_i,\bar{x}_{-i}) \geq \theta_i(\bar{x}_i,\bar{x}_{-i})$" !!!!

Starting point...

Generalized Nash game (GNEP):

$$\boxed{\boxed{\begin{array}{ll}\min\limits_{x_1} & \theta_1(x_1,x_{-1})\\ \text{s.t.} & \{\ x_1\in K_1(x_{-1})\end{array}}\quad\dots\quad\boxed{\begin{array}{ll}\min\limits_{x_n} & \theta_n(x_n,x_{-n})\\ \text{s.t.} & \{\ x_n\in K_n(x_{-n})\end{array}}}$$

$\bar{x}$ is a Generalized Nash Equilibrium
if and only if
for any $i=1,n$, $\bar{x}_i$ is solution of the parametrized problem

$$P_i(\bar{x}_{-i})\qquad \begin{array}{ll}\min_{x_i} & \theta_i(x_i,\bar{x}_{-i})\\ s.t. & x_i\in K_i(\bar{x}_{-i})\end{array}$$

,

Starting point...

Generalized Nash game (GNEP):

$$\boxed{\boxed{\begin{array}{ll} \min\limits_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{\ x_1 \in K_1(x_{-1}) \end{array}} \quad \cdots \quad \boxed{\begin{array}{ll} \min\limits_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{\ x_n \in K_n(x_{-n}) \end{array}}}$$

$\bar{x}$ *is a Generalized Nash Equilibrium if and only if*

$$\bar{x} \in \prod_{i=1}^{n} R_i(\bar{x}_{-i})$$

A very basic example...

Let us consider the following GNEP problem

$$\begin{array}{cc} \min & (y_1 - y_2)^2 \\ s.t. & y_1 \geq y_2 \end{array} \qquad and \qquad \begin{array}{cc} \min & (y_2 - 1)^2 + y_1 \\ s.t. & y_2 \geq 0 \end{array}$$

Then one has

$$R_1(y_2) = \{y_2\} \qquad and \quad R_2(y_1) = 1$$

Thus

$$GNEP = \{(1,1)\}$$

Starting point...

Generalized Nash game (GNEP):

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Now important points to be analysed are:

- Under which conditions an equilibrium exists
- What kinds of first order reformulation
- Qualification conditions (at a point or generically)

But also...

Bilevel problem:

$$\boxed{\begin{array}{ll} \text{" }\min_x\text{ "} & \theta(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} x \in X(y) \\ \min_y \quad \phi(x,y) \\ \quad \text{s.t.} \quad y \in Y(x) \end{array}\right. \end{array}}$$

It would important to study:

- Existence
- First order conditions
- Qualification conditions (at a point or generically)

Problems we want to tackle...

Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{cl} \text{"}\min_x\text{"} & \theta(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} x \in X(y) \\ y \in GNEP(x) \end{array}\right. \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{cl} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \{\ y_1 \in K_1(x,y_{-1}) \end{array}} \quad \dots \quad \boxed{\begin{array}{cl} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \{\ y_n \in K_n(x,y_{-n}) \end{array}}}$$

Problems we could to tackle...

Multi-Leader-Single-Follower-Game (MLSFG):

$$\boxed{\boxed{\begin{array}{ll} \text{"}\min_{x_1}\text{"} & \theta_1(x,y) \\ \text{s.t.} & \begin{cases} x_1 \in X_1(x_{-1},y) \\ y \in S(x) \end{cases} \end{array}} \quad \dots \quad \boxed{\begin{array}{ll} \text{"}\min_{x_p}\text{"} & \theta_p(x,y) \\ \text{s.t.} & \begin{cases} x_p \in X_p(x_{-p},y) \\ y \in S(x) \end{cases} \end{array}}}$$

$$\downarrow\uparrow$$

$$\boxed{\begin{array}{ll} \min_y & \phi(x,y) \\ \text{s.t.} & \left\{\ y \in K(x)\right. \end{array}}$$

Problems we don't want to tackle...yet

Multi-Leader-Multi-Follower-Game (MLFG):

$$\boxed{\boxed{\begin{array}{ll} ''\min_{x_1}'' & \theta_1(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} x_1 \in X_1(x_{-1},y) \\ y \in GNEP(x) \end{array}\right. \end{array}} \quad \cdots \quad \boxed{\begin{array}{ll} ''\min_{x_p}'' & \theta_p(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} x_p \in X_p(x_{-p},y) \\ y \in GNEP(x) \end{array}\right. \end{array}}}$$

$$\downarrow\uparrow \qquad\qquad\qquad \downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \left\{\ y_1 \in K_1(x,y_{-1})\right. \end{array}} \quad \cdots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \left\{\ y_n \in K_n(x,y_{-n})\right. \end{array}}}$$

HUGE ambiguities in MLMF games

If interested, please contact me at:

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SLMF games

Let us first discuss about Single-Leader-Multi-Follower and try to "understand" them:

- *a bit of vocabulary*
- *what kind of interelations between leader and followers?*
- *who are we?*
- *is modelling always "natural" (straightforward)?*

A zoo of SLMFG...

Actually there is a lot of different Single-Leader-Multi-Follower-Game (SLMFG) with different characteristics:

$$\boxed{\begin{array}{ll} \text{"}\min_x\text{"} & \theta_1(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} \color{magenta}{G(x,y)} \leq 0 \\ y \in GNEP(x) \end{array}\right. \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\ldots,y_n,x) \leq 0 \end{array}} \quad \ldots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\ldots,y_n,x) \leq 0 \end{array}}}$$

The constraint $\color{magenta}{G(x,y) \leq 0}$ is called a coupling constraint.

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Then the lower level is a (parametrized) Nash Equilibrium Problem (NEP)

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$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_{sc}(y_1,\ldots,y_n,x) \leq 0 \end{array}} \quad \ldots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_{sc}(y_1,\ldots,y_n,x) \leq 0 \end{array}}}$$

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Thus now the lower level game is a GNEP with a shared constraint!!

Well-posedness

Let us stop and think about the well-posedness of a SLMFG....

Well-posedness

Let us stop and think about the well-posedness of a SLMFG....

....and detect **Ambiguities**

BL = Single-Leader-Single-Follower game

A Bilevel Problem consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \text{“}\min_x\text{”} & F(x,y) \\ s.t. & \left\{\begin{array}{l} x \in X(y) \\ y \in S(x) \end{array}\right. \end{array}$$

where $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_y & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

A trivial example

Consider the following simple bilevel problem

$$\begin{array}{rl} \text{“}\min_{x\in\mathbb{R}}\text{”} & x \\ s.t. & \left\{ \begin{array}{l} x \in [-1,1] \\ y \in S(x) \end{array} \right. \end{array}$$

with $S(x) =$ “y solving

$$\begin{array}{rl} \min_{y\in\mathbb{R}} & -xy \\ s.t & x^2(y^2-1) \leq 0 \text{ ”} \end{array}$$

A trivial example

Lower level problem:

$$\begin{array}{cc} \min_{y\in\mathbb{R}} & -x.y \\ s.t & x^2(y^2-1)\leq 0 \text{ ''} \end{array}$$

Note that the solution map of this convex problem is

$$S(x) := \begin{cases} \{1\} & x<0 \\ \{-1\} & x>0 \\ \mathbb{R} & x=0 \end{cases}$$

Thus for each $x \neq 0$ there is a unique associated solution of the lower level problem

A trivial example

Lower level problem:

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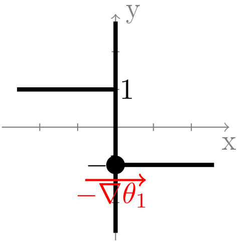
Note that the solution map of this convex problem is



A trivial example

Consider the following simple bilevel problem

$$\text{“}\min_{x\in\mathbb{R}}\text{”} \quad -x.y \qquad s.t. \quad \begin{cases} x \in [-1,1] \\ y \in S(x) \end{cases}$$

with $S(x) =$ given by

$$S(x) := \begin{cases} \{1\} & x < 0 \\ \{-1\} & x > 0 \\ \mathbb{R} & x = 0 \end{cases}$$

First ambiguity...

It means that the problem

$$\boxed{\begin{array}{ll} ''\min_x'' & \theta_1(x,y) \\ \text{s.t.} & \begin{cases} G(x,y) \leq 0 \\ y \in GNEP(x) \end{cases} \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\dots,y_n,x) \leq 0 \end{array}} \quad \dots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\dots,y_n,x) \leq 0 \end{array}}}$$

is **not well posed!!**

Ambiguity: the most simple

And of course the "confortable situation" corresponds to the case of a unique response

$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \left\{\ x \in X(y(x))\right. \end{array}$$

Ambiguity: the most simple

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$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \{\ x \in X(y(x)) \end{array}$$

For example when

$$\textit{for any } x,\ g(x,\cdot) \textit{ is quasiconvex and } f(x,\cdot) \textit{ is strictly convex.}$$

Ambiguity: Optimistic approach

An *Optimistic Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \min_x \min_y & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X(y) \\ y \in S(x) \end{array} \right. \end{array}$$

where $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_y & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

Ambiguity: Pessimistic approach

An *Pessimistic Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \min_x \max_y & F(x,y) \\ s.t. & \begin{cases} x \in X(y) \\ y \in S(x) \end{cases} \end{array}$$

where $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_y & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

Ambiguity: Selection approach

An *"Selection-type" Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{ll} \min_x & F(x, y(x)) \\ s.t. & \left\{ \begin{array}{l} x \in X(y(x)) \\ y(x) \textit{ is a uniquely determined selection of } S(x) \end{array} \right. \end{array}$$

J. Escobar & A. Jofré, *Equilibrium Analysis of Electricity Auctions* (2011)

Ambiguity: The new probabilistic approach

Recently, D.Salas and A. Svensson proposed a **probabilistic approach**:

- *Consider a probability on the different possible follower's reactions*
- *Minimize the expectation of the leader(s)*

Another ambiguity:

What is a solution of a BL/SLMFG?

An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

$$\begin{array}{rl} \min_x \min_y & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

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Instead of considering the previous (optimistic) formulation of BL:

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one can define the (optimistic) value function

$$\varphi_{min}(x) = \min_y \{F(x,y) : g(x,y) \leq 0\} \tag{1}$$

and the BL problem becomes

$$\begin{array}{rl} \min_x & \varphi_{min}(x) \\ s.t. & x \in X \end{array}$$

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Instead of considering the previous (pessimistic) formulation of BL:

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$$\begin{array}{rl} \min_x \max_y & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array}$$

one can define the (pessimistic) value function

$$\varphi_{max}(x) = \max_y \{F(x,y) : g(x,y) \leq 0\} \tag{2}$$

and the Bl problem becomes

$$\begin{array}{rl} \min_x & \varphi_{max}(x) \\ s.t. & x \in X \end{array}$$

An alternative point of view

This is the point of view presented in Stephan Dempe's book:

$$\begin{array}{rl} \min_x \min / \max_y & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array} \quad \textbf{vs} \quad \begin{array}{rl} \min_x & \varphi_{min/max}(x) \\ s.t. & x \in X \end{array}$$

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It immediately raises the question

What is a solution??

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It immediately raises the question

What is a solution??

- *an optimal* x = *leader's optimal strategy?*
- *an optimal couple* (x,y) = *couple of strategies of leader and follower?*

Real life...

Usually when considering SLMFG

$$\begin{array}{rl} \min_x \min_y & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in GNEP(x) \end{cases} \end{array}$$

people says

- *Step A: the leader plays first*
- *Step B: the followers react*

But in real life it's a little bit more complex....

Real life...

Actually in real life, when considering SLMFG

$$\begin{array}{rl} \min_x \min_y & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in GNEP(x) \end{array} \right. \end{array}$$

There are two very different situations:

Real life...

Actually in real life, when considering SLMFG

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There are two very different situations: Indeed

- *Optimal design of the interaction mode: then the "design" (decided by the leader) will be composed of both x and y; In this case one works for the leader* **and** *the followers. Example: Industrial eco-parks*
- *Leader's strategic process: then we work only for the leader Example : demand-side management*

Real life...for decision process

Leader's strategic process:

$$\begin{array}{rl} \min_x \min_y & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in GNEP(x) \end{array} \right. \end{array}$$

We only work for the leader!!

Real life...for decision process

Leader's strategic process:

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We only work for the leader!! Indeed

- *Step 0: the leader has a model of the follower's reaction: optimistic or pessimistic or ...*
- *Step 1: we compute a solution $\bar{x}$ or $(\bar{x}, \bar{y})$ of the SLMFG model*
- *Step 2: the leader plays $\bar{x}$*
- *Step 3: the followers decide to play...*

Real life...for decision process

Leader's strategic process:

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We only work for the leader!! Indeed

- *Step 0: the leader has a model of the follower's reaction: optimistic or pessimistic or ...*
- *Step 1: we compute a solution $\bar{x}$ or $(\bar{x}, \bar{y})$ of the SLMFG model*
- *Step 2: the leader plays $\bar{x}$*
- *Step 3: the followers decide to play...whatever they want!!!*

Another big difficulty...

Consider an Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{ll} \min_x \min_{y_1,\dots,y_n} & \theta_1(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} G(x,y) \le 0 \\ y \in GNEP(x) \end{array}\right. \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\dots,y_n,x) \le 0 \end{array}} \quad \dots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\dots,y_n,x) \le 0 \end{array}}}$$

Another big difficulty...

Actually while modeling a real life application, one can also consider this alternative Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{cl} \min_x \min_{y_1,\dots,y_n} & \theta_1(x,y) \\ \text{s.t.} & y \in GNEP(x) \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{cl} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \leq 0 \\ g_1(y_1,\dots,y_n,x) \leq 0 \end{array} \right. \end{array}} \quad \cdots \quad \boxed{\begin{array}{cl} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \leq 0 \\ g_n(y_1,\dots,y_n,x) \leq 0 \end{array} \right. \end{array}}}$$

Another big difficulty...

Actually while modeling a real life application, one can also consider this alternative Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

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Do these two models generate the same solutions???

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Do these two models generate the same solutions???
NO!!!!!....in general

Back to SLMFG: more precise formulation

Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{cl} \min_x \min_{y_1,\dots,y_n} & \theta(x,y) \\ \text{s.t.} & \left\{\begin{array}{l} G(x,y) \le 0 \\ y \in GNEP(x) \end{array}\right. \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\dots,y_n,x) \le 0 \end{array}} \quad \dots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\dots,y_n,x) \le 0 \end{array}}}$$

All followers are "friends" of the leader!!!

Back to SLMFG: more precise formulation

Pessimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{ll} \min_x \max_{y_1,\dots,y_n} & \theta(x,y) \\ \text{s.t.} & \begin{cases} G(x,y) \leq 0 \\ y \in GNEP(x) \end{cases} \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\dots,y_n,x) \leq 0 \end{array}} \quad \cdots \quad \boxed{\begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\dots,y_n,x) \leq 0 \end{array}}}$$

All followers are "ennemies" of the leader!!!

Back to SLMFG: more precise formulation

Mix Single-Leader-Multi-Follower-Game (SLMFG):

$$\boxed{\begin{array}{cl} \min_x \min_{y_1,\dots,y_p} \max_{y_{p+1},\dots,y_n} & \theta(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \leq 0 \\ y \in GNEP(x) \end{array} \right. \end{array}}$$

$$\downarrow\uparrow$$

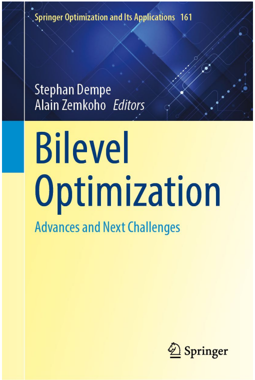
$$\boxed{\boxed{\begin{array}{rl} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\dots,y_n,x) \leq 0 \end{array}} \quad \dots \quad \boxed{\begin{array}{rl} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\dots,y_n,x) \leq 0 \end{array}}}$$

Some are "friends" and some are "ennemies"!!!

Classical formulation: Nash (generalised) game between the follower

An advertisement

A short state of art on Multi-Leader-Follower games, D.A. and A. Svensson, in a book dedicated to Stackelberg, editors A. Zemkoho and S. Dempe, Springer Ed. (2021)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Stephan Dempe
Alain Zemkoho *Editors*

Bilevel Optimization

Advances and Next Challenges

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How to compute?

An SOS1 computation approach to solve Single-Leader-Multi-Follower games

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INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

A tutorial on solving single-leader-multi-follower problems using SOS1 reformulations

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When followers face incertainties

The Radner approach

When followers face incertainties

The Radner approach

- *An existence results for Single-Leader-Radner-Equilibrium game*
- *An application to Industrial Eco-parks*

SLMFG with uncertainties

$$\boxed{\begin{array}{lll} \min\limits_{\substack{x \\ y_1,..,y_p}} & & \theta_1(x,y) \\ & \text{s.t.} & \left\{ \begin{array}{l} x \in X \\ y \in NE(x) \end{array} \right. \end{array}}$$

$$\downarrow\uparrow$$

$$\boxed{\boxed{\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & y_1 \in Y_1(x, y_{-1}) \end{array}} \quad \ldots \quad \boxed{\begin{array}{ll} \min_{y_p} & \phi_p(x,y) \\ \text{s.t.} & y_p \in Y_p(x, y_{-p}) \end{array}}}$$

- Chance constraints in the followers' problems
- Stochastic approach with expectation objective function

Another alternative for SLMFG with uncertainties

Single-Leader-Radner-Equilibrium games are mixtures of bilevel optimization and Radner equilibrium problems.

Another alternative for SLMFG with uncertainties

Single-Leader-Radner-Equilibrium games are mixtures of bilevel optimization and Radner equilibrium problems.

$$\min_{\substack{x \\ y_1,..,y_p}} \quad \theta(x,y) \qquad s.t. \quad \left\{ \begin{array}{l} G(x,y) \leq 0 \\ y \in Radner(x) \end{array} \right.$$

where $Radner(x)$ stands for the Radner equilibriums of the follower game.

Another alternative for SLMFG with uncertainties

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where $Radner(x)$ stands for the Radner equilibriums of the follower game.

E. Allevi, D. A., R. Riccardi, D. Scopelliti,
Single-Leader-Radner-Equilibrium: a new approach for a class of bilevel problems under uncertainty, JOTA (2024)

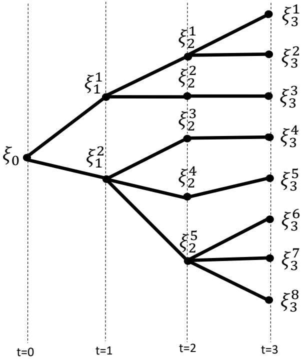
General description of a SLRF game

We consider the following hierarchical structure:

- one of the players acts as the leader of the game;
- a set of non cooperative players, $\mathcal{I} := \{1, \ldots, i, \ldots, I\}$, act as followers of the leader.

The game starts at $t = 0$ and evolves in a sequence of finite future times: the uncertainty is expressed through a finite set of all possible situations that can occur at each time. Let $\mathcal{T} = \{1, \ldots, t, \ldots, T\}$ and $\mathcal{T}_0 = \{0\} \cup \mathcal{T}$ be the finite sets of times

Scenario based approach



The element $\xi_t^{j_t}$ is called *contingency*

The Leader's problem

Leader's variables

for each $i \in \mathcal{I}$, $x_i := (x_{i0}, \ldots, x_{it}, \ldots, x_{iT})$ is a state-contingent vector with

$$x_{it} := (x_{it}(\xi_t^{j_t}))_{\xi_t^{j_t} \in \Xi_t} \quad \text{and} \quad x_{it}(\xi_t^{j_t}) := (x_{it}^h(\xi_t^{j_t}))_{h \in \mathcal{H}} \in \mathbb{R}_+^H,$$

where $x_{it}^h(\xi_t^{j_t})$ represents, for a given time t, the quantity of the h-resource provided by the leader to player i if $\xi_t^{j_t}$ occurs at time t.

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where $x_{it}^h(\xi_t^{j_t})$ represents, for a given time t, the quantity of the h-resource provided by the leader to player i if $\xi_t^{j_t}$ occurs at time t.

Leader's objective function
The objective of the leader is represented by the following function:

$$F : \mathbb{R}^{IHV} \times \mathbb{R}^{IKV} \to \mathbb{R} \quad \text{such that} \quad (x, y) \to F(x, y) \,,$$

where, as we will see in the sequel, $y := (y_1, \ldots, y_i, \ldots, y_I) \in \mathbb{R}_+^{IKV}$ represents the followers' choices on the trade/consumption of the different employments of each resource available at each $\xi_t^{j_t} \in \Xi_0$.

Follower i's problem

Follower i's variables

Let $x \in X$ be the leader decision-vector on the flow-rate of each resource.

Let $\mathcal{K} := \{1, \ldots, k, \ldots, K\}$ be the set of *all the different employments* of all the resources $\mathcal{H}$.

For each $\xi_t^{j_t} \in \Xi_0$ and $h \in \mathcal{H}$, each follower i can use its endowment $x_{it}^h(\xi_t^{j_t})$ in K_h different ways. So, we firstly introduce the state-contingent vector α_i such that, for all $h \in \mathcal{H}$ and $\xi_t^{j_t} \in \Xi_0$, it results:

$$\alpha_{it}^{h,k}(\xi_t^{j_t}) \in [0,1] \quad \textit{and} \quad \sum_{k \in \mathcal{K}_h} \alpha_{it}^{h,k}(\xi_t^{j_t}) = 1.$$

For each resource $h \in \mathcal{H}$ and employment $k \in \mathcal{K}_h$, $\alpha_{it}^{h,k}(\xi_t^{j_t}) x_{it}^h(\xi_t^{j_t})$ represents the percentage of h-resource that player i chooses to use at time t and for employment k if $\xi_t^{j_t}$ occurs

Follower i's problem

- At each time t and when uncertainty $\xi_t^{j_t}$ is revealed, spot market opens for spot trade/consumption; forward market has a financial nature and it opens only at $t = 0$, before the uncertainty is revealed. So, for each $i \in \mathcal{I}$, we introduce the following state-contingent vector:

$$y_i := (y_{i0}, \ldots, y_{it}, \ldots, y_{iT}) \in \mathbb{R}_+^{KV} \quad \textit{such that} \quad y_{it} := (y_{it}(\xi_t^{j_t}))_{\xi_t^{j_t} \in \Xi_t} \in \mathbb{R}_+^{KV_t};$$

$y_{it}^k(\xi_t^{j_t})$ represents the player i spot trade/consumption of the $k \in \mathcal{K}$ at time t if $\xi_t^{j_t}$ occurs.

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$y_{it}^k(\xi_t^{j_t})$ represents the player i spot trade/consumption of the $k \in \mathcal{K}$ at time t if $\xi_t^{j_t}$ occurs.

- Prices $p_t(\xi_t^{j_t}) \in \mathbb{R}_+^K$, where p is the following state-contingent vector of the spot prices:

$$p := (p_0, \dots, p_t, \dots, p_T) \in \mathbb{R}_+^{KV}$$

Follower i's problem

Notice that, at $t = 0$, no one player knows which state will be realised the next time, but it can make plans for each of them: at the beginning of the planning horizon, each player i is allowed to sign contracts of selling or buying a quantity of each $k \in \mathcal{K}$ in the market for the future contingencies, at prices established at $t = 0$. In this way, each player i can transfer such quantities among all future contingencies to have cash for further immediate trade/consumption or for future contracts in next contingencies. Such contracts, called *forward contracts*, with the relative forward prices, are identified in terms of the following state-contingency vectors:

$$z_i := (z_{i1}, \ldots, z_{it}, \ldots, z_{iT}) \in \mathbb{R}^{K(V-1)}, \qquad q := (q_1, \ldots, q_t, \ldots, q_T) \in \mathbb{R}_+^{K(V-}$$

$z_{it}^k(\xi_t^{j_t})$ is the amount of the k-employment traded by player i at $t = 0$ for $\xi_t^{j_t}$ and paid at price $q^k(\xi_t^{j_t})$ at $t = 0$.

Concept of Radner equilibrium

Definition

A Radner equilibrium of plans, prices, and price expectations for $\mathcal{E}_x$ is a vector $(\bar{y}, \bar{z}, \bar{\alpha}, \bar{p}, \bar{q}) \in B\left(x, \bar{\alpha}, \bar{p}, \bar{q}\right) \times \Theta \times \mathbb{R}_+^{KV} \times \mathbb{R}_+^{K(V-1)}$ such that

- for any $i \in \mathcal{I}$:

$$\max_{(y_i, z_i, \alpha_i) \in B_i(x_i, \alpha_i, \bar{p}, \bar{q}) \times \Theta_i} f_i(y_i) = f_i(\bar{y}_i)\,; \tag{3}$$

- for all $\xi_t^{j_t} \in \Xi_0$:

$$\sum_{i \in \mathcal{I}} \bar{y}_{it}^k(\xi_t^{j_t}) \leq \sum_{i \in \mathcal{I}} \bar{e}_{it}^k(\xi_t^{j_t}) \qquad \forall k \in \mathcal{K}; \tag{4}$$

- for all $\xi_t^{j_t} \in \Xi$:

$$\sum_{i \in \mathcal{I}} \bar{z}_{it}^k(\xi_t^{j_t}) = 0 \qquad \forall k \in \mathcal{K}\,. \tag{5}$$

It could be also called a Bayesian approach with incomplete information

Single-Leader-Radner-Equilibrium

Definition

A vector $(\bar{x}, \bar{y}, \bar{z}, \bar{\alpha}, \bar{p}, \bar{q}) \in X \times B\,(\bar{x}, \bar{\alpha}, \bar{p}, \bar{q}) \times \Theta \times \mathbb{R}_+^{KV} \times \mathbb{R}_+^{K(V-1)}$ is an *optimistic solution* of the Single-Leader-Radner-Equilibrium game if it solves the following problem:

$$\begin{aligned} &\min_{x,y,z,\alpha,p,q} F(x,y) \\ &\text{s.t.} \quad \begin{cases} x \in X \\ (y,z,\alpha,p,q) \in RE(\mathcal{E}_x)\,. \end{cases} \end{aligned} \tag{6}$$

Basic assumptions

Assumptions A
For any $i \in \mathcal{I}$*, one has:*
(A.1) f_i *is continuous and quasiconcave;*
(A.2) f_i *is strictly increasing in the trade/consumption of* $\hat{k} \in \mathcal{K}_{\hat{h}}$ *that is*

$$\forall \tilde{y}_i,\ \tilde{\tilde{y}}_i \in \mathbb{R}_+^{KV} \ \ \textit{such that} \ \ \tilde{y}_i \geq \tilde{\tilde{y}}_i \quad \Rightarrow \quad f_i(\tilde{y}_i) > f_i(\tilde{\tilde{y}}_i),$$

where $\tilde{y}_i \geq \tilde{\tilde{y}}_i$ means that there exists $\xi_t^{j_t} \in \Xi_0$ with $\tilde{y}_{it}^{\hat{k}}(\xi_t^{j_t}) > \tilde{\tilde{y}}_{it}^{\hat{k}}(\xi_t^{j_t})$.

Assumption (A.2) expresses that player i has preferences expressed by means of a utility function f_i that is strictly increasing in the trade/consumption of the $\hat{k}$-employment of the resource $\hat{h}$ of which it is endowed from the leader of a positive quantity $x_{it}^{\hat{h}}(\xi_t^{j_t})$.

Reformulation of Radner equilibrium

We proved a quasi-variational inequality reformulation of the lower level equilibrium problem $\mathcal{E}_x$ of Single-Leader-Radner-Equilibrium problem

Theorem

Let $x \in X$ *be fixed and Assumptions A be satisfied.*
The vector $(\bar{y}, \bar{z}, \bar{\alpha}, \bar{p}, \bar{q}) \in B(x, \bar{\alpha}, \bar{p}, \bar{q}) \times \Theta \times \Delta$ *is an equilibrium of plans, prices, and price expectations for* $\mathcal{E}_x$
if and only if
it is solution to the quasi-variational inequality problem

$$\text{Find } (\bar{y}, \bar{z}, \bar{\alpha}, \bar{p}, \bar{q}) \in B(x, \bar{\alpha}, \bar{p}, \bar{q}) \times \Theta \times \Delta \text{ such that}$$
$$\exists\, h := (h_i)_{i \in \mathcal{I}} \in G(\bar{y}) \text{ satisfying}$$

$$\sum_{i \in \mathcal{I}} \langle h_i, y_i - \bar{y}_i \rangle_{KV} + \langle (\sum_{i \in \mathcal{I}} (\bar{e}_i - \bar{y}_i), -\sum_{i \in \mathcal{I}} \bar{z}_i), (p, q) - (\bar{p}, \bar{q}) \rangle_{K(2V-1)} \geq 0$$

$$\forall\ (y, z, \alpha, p, q) \in B(x, \bar{\alpha}, \bar{p}, \bar{q}) \times \Theta \times \Delta.$$

Quasiconvexity

- A function $f : X \to \mathbb{R}_\infty$ is said to be *quasiconvex* on K if,

for all x, $y \in K$ *and all* $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$

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or

for all $\lambda \in \mathbb{R}$, *the sublevel set*

$$S_\lambda = \{x \in X \ : \ f(x) \le \lambda\} \textit{ is convex.}$$

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f *differentiable*

$$f \text{ is quasiconvex} \iff df \text{ is quasimonotone}$$

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or

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or

$$f \text{ is quasiconvex} \iff \partial f \text{ is quasimonotone}$$

Quasiconvexity

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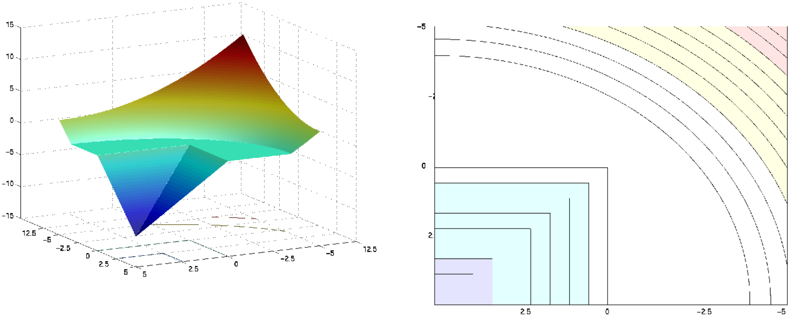
for all $\lambda \in \mathbb{R}$, *the sublevel set*

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \text{ is convex.}$$

- A function $f : X \to \mathbb{R}_\infty$ is said to be *semistrictly quasiconvex* on K if, f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall\, z \in [x, y[.$$



convex ⇒ semistrictly quasiconvex ⇒ quasiconvex

Adjusted sublevel set:

For any $x \in \mathrm{dom} f$, we define

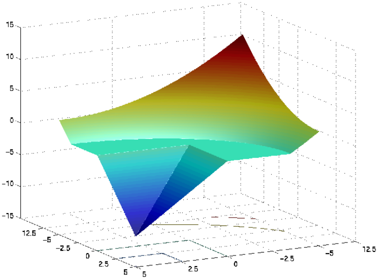
$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S^<_{f(x)}, \rho_x)$$

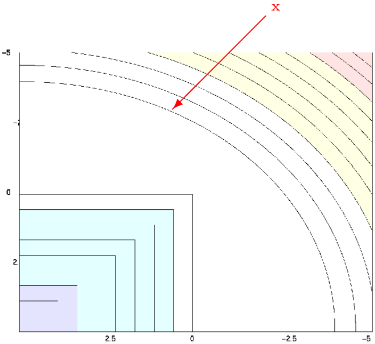
where $\rho_x = dist(x, S^<_{f(x)})$, if $S^<_{f(x)} \neq \emptyset$.

Ajusted normal operator:

$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

Example





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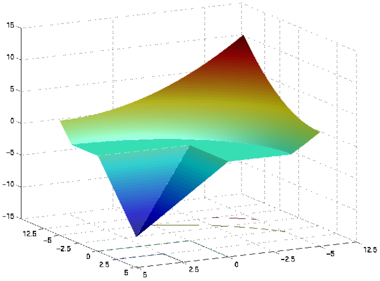
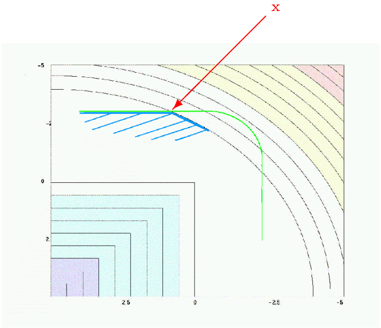
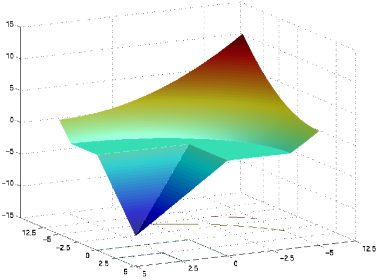


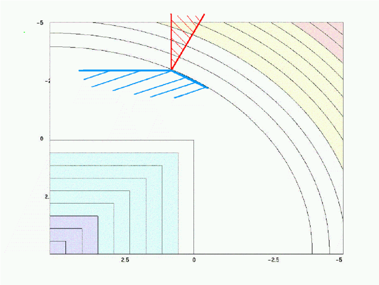


$$\overline{B}(S^{<}_{f(x)}, \rho_x)$$

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$$

Example





$$S_f^a(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$$

$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

Normal operator: properties

A sufficient optimality condition

Proposition

Let $f : \mathbb{R}^p \to \mathbb{R}$ be a quasiconvex and sub-boundarily constant function, and $C \subset \mathbb{R}^p$ be a nonempty set. Then, any solution of the Stampacchia variational inequality defined by the operator $N_f^a \setminus \{0\}$ on C is a global minimizer of f over C, that is

$$QVI\left(N_f^a \setminus \{0\}, C\right) \subset \arg\min_C f.$$

- *D. Aussel & N. Hadjisavvas*, Adjusted sublevel sets, normal operator and quasiconvex programming, *SIAM J. Optim., 16 (2005), 358-367.*
- *D. Aussel, D. Salas, K. Cao Van, Existence results for generalized Nash equilibrium problems under continuity-like properties of sublevel sets., SIAM J. Optim. (2019), Vol. 29, No. 2, pp. 1558-1577.*

OPQVIC reformulation

Theorem

Let Assumptions A be satisfied. A vector $(\bar{x},\bar{y},\bar{z},\bar{\alpha},\bar{p},\bar{q}) \in X \times B(\bar{x},\bar{\alpha},\bar{p},\bar{q}) \times \Theta \times \mathbb{R}_+^{KV} \times \mathbb{R}_+^{K(V-1)}$ *is solution to the SLRE if and only it is solution to the* optimization problem with quasi-variational inequality constraint

$$OPQVIC(F,X,QVI(\widetilde{\Phi},\widetilde{\Gamma})) \qquad \min_{x,y,z,\alpha,p,q} F(x,y)$$

$$s.t. \quad \begin{cases} x \in X \\ (y,z,\alpha,p,q) \in QVI(\widetilde{\Phi}(x),\widetilde{\Gamma}(x)) \end{cases},$$

where

- the map $\widetilde{\Phi} : X \times \mathbb{R}_+^{IKV} \times \mathbb{R}^{IK(V-1)} \times \Theta \rightrightarrows \mathbb{R}^{IKV} \times \mathbb{R}^{KV} \times \mathbb{R}^{K(V-1)}$ such that

$$\widetilde{\Phi}(x,y,z,\alpha) := \widetilde{\Phi}(x) = \left(G(y), \sum_{i\in\mathcal{I}} (y_i - e_i), \sum_{i\in\mathcal{I}} z_i \right) ;$$

- the map $\widetilde{\Gamma} : X \times \mathbb{R}_+^{KV} \times \mathbb{R}_+^{K(V-1)} \times \Theta \times \Delta \rightrightarrows \mathbb{R}_+^{IKV} \times \mathbb{R}^{IK(V-1)} \times \Theta \times \Delta$ such that

$$\widetilde{\Gamma}(x,\alpha,p,q) := \widetilde{\Gamma}(x) = (B(x,\alpha,p,q),\Theta,\Delta) .$$

Proposition

Let Assumptions A be satisfied. A vector $(\bar{x}, \bar{y}, \bar{z}, \bar{\alpha}, \bar{p}, \bar{q}) \in X \times B(\bar{x}, \bar{\alpha}, \bar{p}, \bar{q}) \times \Theta \times \mathbb{R}_+^{KV} \times \mathbb{R}_+^{K(V-1)}$ *is solution to the SLRE* (6) *if and only it is solution to the* optimization problem with quasi-variational inequality constraint

$$OPQVIC(F, X, QVI(\widetilde{\Phi}, \widetilde{\Gamma})) \qquad \min_{x,y,z,\alpha,p,q} F(x,y)$$

$$s.t. \quad \begin{cases} x \in X \\ (y, z, \alpha, p, q) \in QVI(\widetilde{\Phi}(x), \widetilde{\Gamma}(x)) \,, \end{cases}$$

where, for any $x \in X$, $QVI(\widetilde{\Phi}(x), \widetilde{\Gamma}(x))$ *represents the solution set of the parametric quasi-variational inequality problem with the set-valued maps defined by*

- *the map* $\widetilde{\Phi} : X \times \mathbb{R}_+^{IKV} \times \mathbb{R}^{IK(V-1)} \times \Theta \rightrightarrows \mathbb{R}^{IKV} \times \mathbb{R}^{KV} \times \mathbb{R}^{K(V-1)}$ *such that*

$$\widetilde{\Phi}(x, y, z, \alpha) := \widetilde{\Phi}(x) = \left(G(y), \sum_{i \in \mathcal{I}} (y_i - e_i), \sum_{i \in \mathcal{I}} z_i \right) \;;$$

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Existence result

Assumptions A
For any $i \in \mathcal{I}$, one has:
(A.1) f_i is continuous and quasiconcave;
(A.2) f_i is strictly increasing in the trade/consumption of $\hat{k} \in \mathcal{K}_{\hat{h}}$

Theorem

Let F be lower semicontinuous and Assumptions A be satisfied. Then, the $SLRE(F, X, S)$ problem admits at least a solution.

When followers face incertainties

The Radner approach

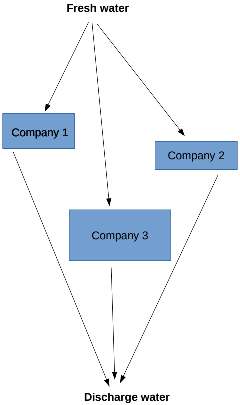
An application to Industrial Eco-parks

What is an « Industrial Eco-park » ?

Example of water management

- In a geographical area, there are different companies $1, \ldots, n$
- Each of them is buying fresh water (high price) for their production processes
- Each company generates some "dirty water" and have to pay for discharge

Stand alone situation



How does it work ?

The aims in designing Industrial Eco-park (IEP) are

a) to reduce the cost of production of each company

b) to reduce the environmental impact of the whole production

Thus "Eco" of IEP is at the same time Economical and ecological

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Thus "Eco" of IEP is at the same time Economical and ecological

Keystone: share/reuse of ressources.....so simple!!!

What is an « Eco-park » ?

Example of water management

How to reach these aims?

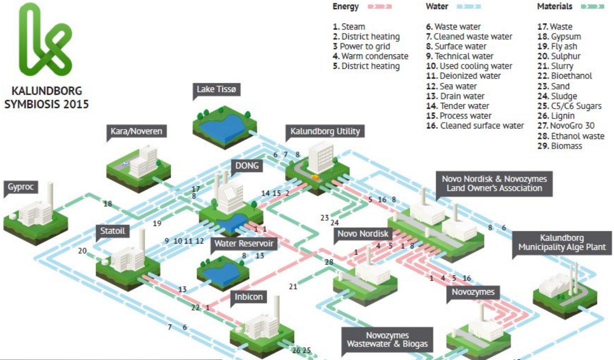
a) create a network (water tubes) between the companies

b) Eventually install some regeneration unit (cleaning of the water)

It is important to understand that this approach is not limited to water. It can be applied to vapor, gas, coaling fluids, human resources...

Kalundborg (Danemark)

An symbolic example of Industrial eco-park is Kalundborg (Danemark)



Definition

What model for an « Industrial Eco-park » ?

In order to convince companies to participate to the Ecopark, our model should guarantee that:

a) **each company** will have a lower cost of production in Eco-park organization than in stand-alone organization

b) the Eco-park organization must generate a **lower freshwater consumption** than with a stand-alone organization

MOO classical treatment

Since the 60', Eco-park design was modelled through Multi-objective Optimization by the evaluation of Pareto fronts (Gold programming algorithms, scalarization...).

$$\min \quad \left\{ \begin{array}{l} \textit{Fresh water consumption} \\ \textit{Individual costs of producer } 1 \\ \qquad\qquad \vdots \\ \textit{Individual costs of producer } n \end{array} \right.$$

$$s.t. \quad \left\{ \begin{array}{l} \textit{Water balances} \\ \textit{Topological constraints} \\ \textit{Water quality criteria} \end{array} \right.$$

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And then usually the selection of a Pareto point is done (which one? min distance to utopia....)

Alternative approach of IEP

The needed change :

. . . to have an independant designer/regulator

. . . to have fair solutions for the companies

Thus we propose to use a mixture between two models:

- Hierarchical optimisation (bi-level optim.)
- Nash game concept between the companies

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Thus we propose to use a mixture between two models:

- Hierarchical optimisation (bi-level optim.)
- Nash game concept between the companies

Thus leading to a **Multi-Leader-Follower approach!!**

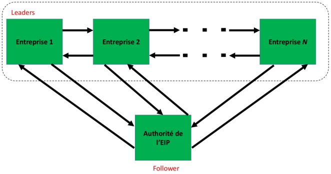
Two possible models

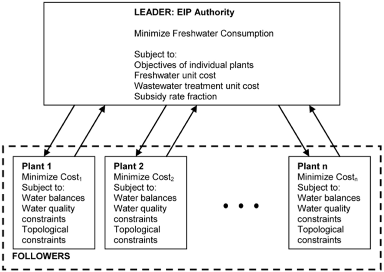
The needed change :

…to have an independant designer/regulator

Using the concepts of Nash equilibrium, we propose two different possible models:

- Multi-Leader-Common-Follower game (MLCFG)
- Single-Leader-Multi-Follower game (SLMFG)





LEADER: EIP Authority

Minimize Freshwater Consumption

Subject to:
Objectives of individual plants
Freshwater unit cost
Wastewater treatment unit cost
Subsidy rate fraction

Plant 1
Minimize Cost$_1$
Subject to:
Water balances
Water quality constraints
Topological constraints

Plant 2
Minimize Cost$_2$
Subject to:
Water balances
Water quality constraints
Topological constraints

• • •

Plant n
Minimize Cost$_n$
Subject to:
Water balances
Water quality constraints
Topological constraints

FOLLOWERS

Some results

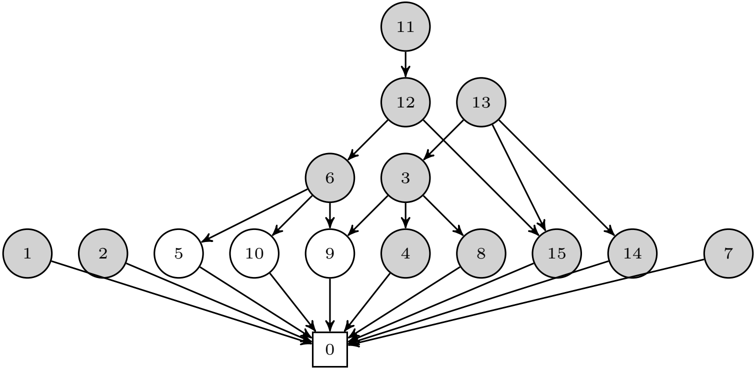
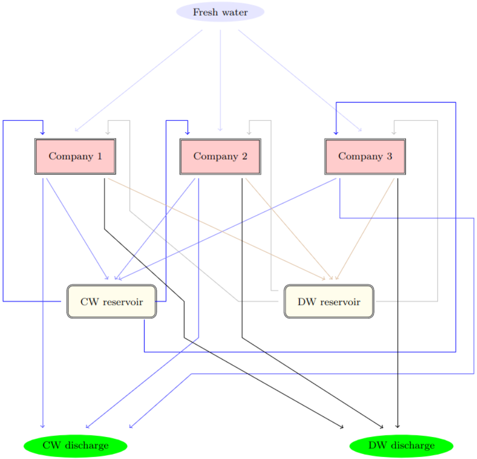


Figure: The configuration in the case without regeneration units, $\alpha_i = 0.95$. Gray nodes are consuming strictly positive fresh water.

An Eco-park with uncertainties

Water management process



Uncertainties on the level of production of the companies

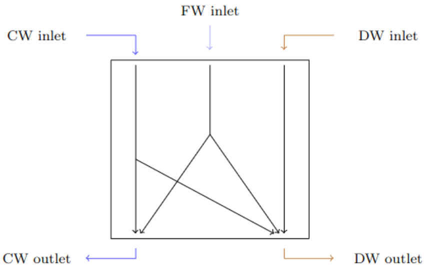
Leader: local government or manager of IEP

Variables of the leader: quantity of FW provided by the leader to company i

Followers: 3 companies having one process using water (with certain tolerance on the "pollution")

Followers' variables:

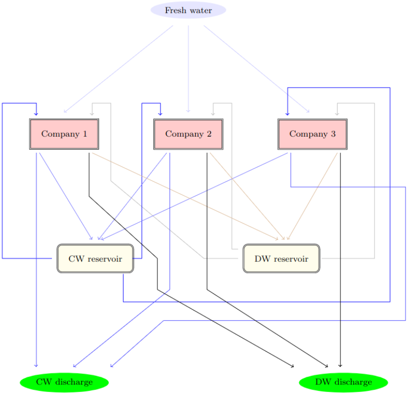
- the company i decision on the optimal transformation of FW, CW, and DW in its industrial process, in terms of CW and DW: $CW \rightarrow CW, DW$, $FW \rightarrow CW, DW$...
- the quantity of CW and DW bought by company i in the markets after the uncertainty is revealed (contracts);



And for the EIP under uncertainties...

Corollary

The optimal design of the eco-industrial park admits an optimistic solution.



SLMF games without partial coalitions

On going work!

NEP/GNEP...but

A point $\bar{x}$ is generalized equilibrium if no player can unilaterally deviate from $\bar{x}$ without "losing".

- The concept of Nash equilibrium accounts only for unilateral deviations
- Thus it excludes the possibility of coordination among followers/players.

NEP/GNEP...but

A point $\bar{x}$ is generalized equilibrium if no player can unilaterally deviate from $\bar{x}$ without "losing".

- The concept of Nash equilibrium accounts only for unilateral deviations
- Thus it excludes the possibility of coordination among followers/players.

Consequently, a Nash equilibrium may be unstable if a group of followers/players can jointly deviate and improve their outcomes.

Strong Nash equilibrium

The concept of *strong Nash equilibrium*, introduced by Aumann[1], addresses this limitation concerning coalition formation.

A strong Nash equilibrium (SNE) is a strategy profile from which no coalition of players can cooperatively deviate in a way that strictly benefits all of its members, given that the actions of the other players remain fixed.

Strong Nash equilibrium

The concept of *strong Nash equilibrium*, introduced by Aumann[1], addresses this limitation concerning coalition formation.

A strong Nash equilibrium (SNE) is a strategy profile from which no coalition of players can cooperatively deviate in a way that strictly benefits all of its members, given that the actions of the other players remain fixed.

Hence, the concept of SNE is robust not only to individual deviations but also to coordinated deviations by coalitions.

R. Aumann (1959), Acceptable points in general cooperative n-person games in "Contributions to the Theory of Games IV", Princeton Univ. Press, Princeton, N.J.

Strong solution of a NEP

Suppose

- $\{Y_i|\, i \in \mathcal{I}\}$ is a collection of Hausdorff topological vector spaces,
- $D_i \subseteq Y_i$ denotes the strategy spaces of player $i \in \mathcal{I}$,
- $\Theta_i : Y_i \times Y_{-i} \to \mathbb{R}$ denotes the objective function of player i

then a point $\bar{y} \in D = \prod_{i\in\mathcal{I}} D_i$ is said to be a strong generalized Nash equilibrium if there does not exist any $\mathcal{C} \subseteq \mathcal{I}$ and $z_{\mathcal{C}} = (z_i)_{i\in\mathcal{C}} \in \prod_{i\in\mathcal{C}} D_i$ such that

$$\Theta_i(z_{\mathcal{C}}, \bar{y}_{-\mathcal{C}}) > \Theta_i(\bar{y}) \textit{ for all } i \in \mathcal{C}. \qquad (7)$$

Some recent works

- *V. Scalzo, Existence of strong equilibrium in discontinuous games, J. Math. Anal. Appl. 491(1) (2020).*
- *V. Scalzo, Existence of doubly strong equilibria in generalized games and quasi-Ky Fan minimax inequalities, J. Math. Anal. Appl. 514 (2022).*

One can easily extend this concept to the case where each player has a feasible strategy map $K_i : D_{-i} \rightrightarrows D_i$ that depends on the strategies of the other players.

Suppose

- $\{Y_i | i \in \mathcal{I}\}$ *is a collection of Hausdorff topological vector spaces,*
- $D_i \subseteq Y_i$ *denotes the strategy spaces of player* $i \in \mathcal{I}$,
- $\Theta_i : Y_i \times Y_{-i} \to \mathbb{R}$ *denotes the objective function of player* i
- $K : D \rightrightarrows D$ *defined as* $K(x) = \prod_{i \in \mathcal{I}} K_i(x_{-i})$ *denotes the product of feasible strategy maps.*

Then, any point $\bar{y} \in K(\bar{y})$ *is said to be strong generalized Nash equilibrium if there does not exist any* $\mathcal{C} \subseteq \mathcal{I}$ *and* $z_{\mathcal{C}} = (z_i)_{i \in \mathcal{C}} \in \prod_{i \in \mathcal{C}} K_i(\bar{y}_{-i})$ *such that*

$$\Theta_i(z_{\mathcal{C}}, \bar{y}_{-\mathcal{C}}) > \Theta_i(\bar{y}) \text{ for all } i \in \mathcal{C}. \tag{8}$$

Single-Leader-Strong-Equilibrium game

Suppose now that we consider a Single-Leader-Multi-Follower game.

Any vector $(\bar{x}, \bar{y}) \in X \times \prod_{i \in \mathcal{I}} (K_i(\bar{y}_{-i}), \bar{x}))$ *is known as an optimistic solution for the Single-Leader-Strong-Equilibrium game if it solves the following problem:*

$$\min_{x,y} F(x,y) \text{ such that } \begin{cases} x \in X \\ y \in S(x). \end{cases}$$

where $S(x)$ denotes the set of all strong generalized Nash equilibrium for follower's problem subject to leader's decision variable x.

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Our first aim: conditions for existence...

Theorem

Let $G(x) = (Y_i, K_i(\cdot, x), \Theta_i(\cdot, x))_{i\in\mathcal{I}}$ *be the parametrized generalized game with respect to* $x \in X$*. Suppose that*

(a) X *is compact and* F *is lower semi-continuous;*

(b) $K_i(y, \cdot) : X \rightrightarrows Y_i$ *is lower semi-continuous;*

For a given leader's variable $x \in X$*,*

(c) $G(x)$ *satisfies the SE-generalized deviation property;*

(d) $G(x)$ *is SE-uniformly quasi-concave;*

(e) $K_i(\cdot, x) : Y \rightrightarrows Y_i$ *is upper semi-continuous with compact values;*
For any follower $i \in \mathcal{I}$ *and followers' strategy* $y \in Y$*:*

(e) $\Theta_i(y, \cdot) : X \to \mathbb{R}$ *is pseudo-continuous.*

Then, the Single-Leader-Strong-Equilibrium game admits a solution.

SE-generalized deviation property

For the given leader's variable x and strategy $z \in Y$, we define $M_i(z, x) = \{y \in X \,|\, \Theta_i(y) > \Theta_i(z)\}$.

Definition

A follower's GNEP $G(x) = (Y_i, K_i(\cdot, x), \Theta_i(\cdot, x))_{i\in\mathcal{I}}$ satisfies the *SE-generalized deviation property* for the given leader's variable x if $z \notin S(x)$ implies that there exists an open nbd O_z of z in Y and an upper semi-continuous map $\zeta_z : O_z \times X \rightrightarrows Y$ with non-empty convex compact values such that for all $z' \in O_z$

(i) $\zeta_z(z', x) \subseteq K(z', x)$;

(ii) for all $y' \in \zeta_z(z', x)$ there exists a coalition C such that $(y'_C, z'_{-C}) \in \cap_{i\in C} M_i(z', x)$.

SE-Uniformly quasi-concavity

Definition

A follower's GNEP $G = (Y_i, K_i(\cdot, x), \Theta_i(\cdot, x))_{i\in\mathcal{I}}$ is *SE-uniformly quasi-concave* if for all $\{y_1, \cdots, y_k\} \subset Y$, and $z \in conv\{y^{h_1}, \cdots, y^{h_l}\}$ implies there exists $y \in \{y^{h_1}, \cdots, y^{h_l}\}$ such that for any $\mathcal{C} \subseteq \mathcal{I}$, there exists $i \in \mathcal{C}$ such that $\Theta_i(y_{\mathcal{C}}, z_{-\mathcal{C}}, x) \leq \Theta_i(z, x)$.

Theorem

For any leader's variable $x \in X$, *let* $G(x) = (Y_i, K_i(\cdot, x), \Theta_i(\cdot, x))_{i \in \mathcal{I}}$ *satisfies the SE-generalized deviation property and SE-uniformly quasi-concavity. Then, the set of strong generalized Nash equilibrium is non-empty.*

QVI reformulation

Let U be a Hausdorff topological vector space, $X \subseteq U$ be non-empty and $K : Y \times X \rightrightarrows Y$ be set-valued map.

QVI reformulation

Let U be a Hausdorff topological vector space, $X \subseteq U$ be non-empty and $K : Y \times X \rightrightarrows Y$ be set-valued map.

Define $\Phi : Y \times Y \times X \to \mathbb{R}$ *as,*

$$\Phi(y,z,x) = \sum_{C \in \mathcal{I}} \max_{i \in C} \Phi_i^C(y,z,x) \textit{ for any } (y,z,x) \in Y \times Y \times X,$$

$$\textit{where } \Phi_i^C(y,z,x) = \begin{cases} 0, & \textit{if } \theta_i(y_C, y_{-C}, x) \geq \theta_i(z_C, y_{-C}, x) \\ -1, & \textit{if } \theta_i(y_C, y_{-C}, x) < \theta_i(z_C, y_{-C}, x). \end{cases}$$

Then, we consider the quasi-equilibrium problem QEP(Φ, K) which intends to find $\bar{y} \in K(\bar{y}, x)$ such that

$$\Phi(\bar{y}, z, x) \geq 0, \quad \forall\, z \in K(\bar{y}, x).$$

Proposition

Suppose $x \in X$ *is arbitrary. Then,* $\bar{y}$ *is a solution for* $QEP(\Phi(\cdot,\cdot,x), K(\cdot,x))$ *if and only if it is a generalized strong Nash equilibrium for* $G(x)$.

Scalzo 2022-Theorem 2

Theorem

Assume that X *and* Y *are non-empty convex compact and for any* $x \in X$*:*

(a) *the map* $G(x)$ *is SE-uniformly quasi-concave;*

(b) $K(\cdot, x) : Y \rightrightarrows Y$ *is upper semi-continuous with non-empty convex and compact values;*

(c) *the map* $F(\cdot, x) : Y \rightrightarrows Y$ *defined as* $F(z, x) = \{y \in K(z) \,|\, \Phi(y, z, x) > 0\}$ *admits generalized open lower sections.*

Then the parametrized quasi-equilibrium problem $QEP(\Phi(x, \cdot), K(x, \cdot))$ *admits a solution for any* $x \in X$*.*

Definition

Let $\mathcal{U}$ and $\mathcal{V}$ be topological spaces. A map $F : \mathcal{U} \rightrightarrows \mathcal{V}$ is said to be generalized open lower sections if $F(z) \neq \emptyset$ implies that there exists an open nbd O_z of z and a well-behaved map $\zeta_z : O_z \rightrightarrows \mathcal{V}$ such that $\zeta(z') \subseteq F(z')$ for all $z' \in O_z$.

Proposition

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For any follower $i \in \mathcal{I}$ *and followers' strategy* $y \in Y$:

(e) $\Theta_i(y, \cdot) : X \to \mathbb{R}$ *is pseudo-continuous.*

Then, the solution map $S : X \rightrightarrows Y$ *defined as solution map for follower's problem is upper semi-continuous with non-empty and compact values.*

SLMF with nonself constraint maps

SLMF with nonself constraint maps

For next webminar....

Take away

SLMF can (should/must) be considered **with different types of interactions between the followers***:*

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- *with local (generalised) Nash equilibrium D. A. and P. Chaipunya, JOTA (2024)*
- *without utility functions (preference equilibrium)*
  D. Aussel, M. Guili, M. Milasi and D. Scopelliti, SIAM J. Optim. (2025) and preprint (2026)
- ...

Thanks you very much

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